Boundary Conditions for the Macroscopic Slip Velocity on a Curved Wall

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Equations for the macroscopic slip velocity on curved walls are derived using the Chapman-Enskog solution of Boltzmann's equation.

1. In order to handle the slip velocity problem and to gain the value of the macroscopic slip velocity, the Boltzmann equation has to be solved with the boundary conditions depending on gas-surface-interactions. This problem was solved by Loyalka [1] for arbitrary gas-wall-interactions and plane surfaces.

In the elementary treatment of the Kramer's problem [2], it is assumed that the velocity distribution of gas molecules impinging on the wall is equal to the extrapolation of the distribution function far from the surface. It follows that the slip boundary condition has the same form as that gained by the exact solution of Boltzmann's equation in the neighborhood of the boundary. Only the slip coefficients in the two treatments show somewhat different numerical values. Therefore, it may be guessed that the form of the slip boundary conditions for arbitrary curved walls can be derived using the first approximation of the Chapman-Enskog solution. The results of this procedure are given by the Equations (15), (16).

2. We start from the definition of the accommodation coefficient σ of the momentum tangential component

$$\boldsymbol{\tau} = \sigma \, \boldsymbol{\tau}_{\mathrm{i}} \, . \tag{1}$$

$\boldsymbol{\tau}$ denotes the shear stress of the gas at the wall, i. e. the normal component of the momentum flux tangential component of the gas molecules.

$$\boldsymbol{\tau} = \boldsymbol{n} \cdot \int \boldsymbol{v} \, m \, \boldsymbol{v}_{\mathrm{t}} \, f(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \, . \tag{2}$$

Here m is the mass of a gas molecule, $\boldsymbol{v}$ its velocity, $f(\boldsymbol{v})$ the velocity distribution function, and $\boldsymbol{n}$ the (outwards directed) normal unit vector of the wall. The normal component of velocity is

$$v_{\mathrm{n}} = \boldsymbol{v} \cdot \boldsymbol{n} \, ,$$

its tangential component

$$\boldsymbol{v}_{\mathrm{t}} = \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n}) \, .$$

$\boldsymbol{\tau}_{\mathrm{i}}$ in (1) is the normal component of the momentum flux tangential component of the gas molecules impinging on the wall,

$$\boldsymbol{\tau}_{\mathrm{i}} = \boldsymbol{n} \cdot \int_{\boldsymbol{n} \cdot \boldsymbol{v} > 0} \boldsymbol{v} \, m \, \boldsymbol{v}_{\mathrm{t}} \, f(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \, ,$$

written in that coordinate system in which the velocity of the solid surface vanishes. A consideration of the case where $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_{\mathrm{i}}$ have not the same direction, i. e. where a matrix of accommodation coefficients occurs, shows the following. If the wall is isotropic concerning the tangential momentum transfer; and if the coefficients of the accommodation matrix are material constants independent on the curvature of the wall and on the gas flow, the resulting slip boundary conditions are equal to those derived from Equation (1).

The mass velodity $\boldsymbol{u}$ of the gas molecules,

$$\boldsymbol{u} = (1/n(\boldsymbol{r})) \int \boldsymbol{v} \, f(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \, , \tag{3}$$

has the following boundary condition at the wall.

$$u_n = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \, . \tag{4}$$

$n(\boldsymbol{r})$ in (3) is the particle density of the gas molecules at the point $\boldsymbol{r}$. With the boundary condition (4), the shear stress $\boldsymbol{\tau}$ can be given as a function of the pressure tensor $\mathbf{P}$ of the gas

$$\mathbf{P} = \int m \, \mathbf{V} \, \mathbf{V} \, f(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \, ,$$

where

$$\mathbf{V} = \boldsymbol{v} - \boldsymbol{u}$$

is the peculiar velocity of a gas molecule. From (2), (3), (4) it follows that

$$\boldsymbol{\tau} = \boldsymbol{n} \cdot \mathbf{P} \cdot \left(\frac{\boldsymbol{u}}{u} \frac{\boldsymbol{u}}{u} + \frac{\boldsymbol{u} \times \boldsymbol{n}}{u} \frac{\boldsymbol{u} \times \boldsymbol{n}}{u} \right) . \tag{5}$$

The expression for $\boldsymbol{\tau}_{\mathrm{i}}$ corresponding to (5), is calculated using the first approximation of the Chap-

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man-Enskog solution. In the absence of a temperature gradient latter is given by [3]

$$f(\boldsymbol{v}) = f^{(0)}(\boldsymbol{v}) + f^{(1)}(\boldsymbol{v}) \,, \tag{6}$$

$$f^{(0)}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = \pi^{-3/2}\, e^{-U^2}\, n(\boldsymbol{r})\,\mathrm{d}\boldsymbol{U} \,, \tag{7}$$

$$f^{(1)}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = -\pi^{-3/2}\, e^{-U^2}\, 2\,\mathbf{B}\,..\,\nabla\boldsymbol{u}\,\mathrm{d}\boldsymbol{U} \,, \tag{8}$$

where

$$\boldsymbol{U} = \mathbf{V}\sqrt{m/2\,k\,T} \,, \tag{9}$$

and

$$\mathbf{B} = \overset{\circ}{\boldsymbol{U}\boldsymbol{U}}\, B(U) \,. \tag{10}$$

With (4) and (6) to (10) the equation for $\boldsymbol{\tau}_\mathrm{i}$ runs

$$\boldsymbol{\tau}_\mathrm{i} = \tfrac{1}{2}\,\boldsymbol{n}\cdot\mathbf{P}\cdot\left(\frac{\boldsymbol{u}}{u}\frac{\boldsymbol{u}}{u} + \frac{\boldsymbol{u}\times\boldsymbol{n}}{n}\frac{\boldsymbol{u}\times\boldsymbol{n}}{u}\right) + m\,u\,(\tfrac{1}{4}\,n(\boldsymbol{r})\,\overline{V} + \int_{v_n>0} V_n\, f^{(1)}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v})\,\boldsymbol{u}/u \,, \tag{11}$$

with the arithmetic mean speed

$$\overline{V} = (2/\sqrt{\pi})\,\sqrt{2\,k\,T/m} \,.$$

(5) and (11) are introduced to Equation (1). This yields the following two equations:

$$u = \frac{(2-\sigma)\,n\cdot\mathbf{P}\cdot\boldsymbol{u}/u}{2\,\sigma\, m\,(\tfrac{1}{4}\,n(\boldsymbol{r})\,V + \int_{v_n>0} V_n\, f^{(1)}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v})} \,, \tag{12}$$

$$\boldsymbol{n}\cdot\mathbf{P}\cdot(\boldsymbol{u}\times\boldsymbol{n})/u = 0 \,. \tag{13}$$

The Chapman-Enskog solution (6) to (10) yields as an equation for the stress tensor in the first approximation

$$\mathbf{P} = p\,\mathbf{1} - 2\,\mu\,\overline{\overline{\nabla\boldsymbol{u}}}^{\,0} \,,$$

with p as hydrostatic pressure, and μ as viscosity.

Hence, with Eq. (4) the boundary conditions (12), (13) up to terms of the first order turn to

$$u = -\zeta\,((\nabla\boldsymbol{u})\cdot\boldsymbol{n} + \boldsymbol{n}\cdot\nabla\boldsymbol{u})\cdot\frac{\boldsymbol{u}}{u} \,, \tag{14}$$

$$((\nabla\boldsymbol{u})\cdot\boldsymbol{n} + \boldsymbol{n}\cdot\nabla\boldsymbol{u})\cdot\frac{\boldsymbol{u}\times\boldsymbol{n}}{u} = 0 \,. \tag{15}$$

Herein ζ denotes the slip coefficient,

$$\zeta = \frac{2-\sigma}{\sigma}\,\frac{2\,\mu}{m\,n(\boldsymbol{r})\,\overline{V}} \,.$$

The physical meaning of the boundary conditions (14), (15) is illustrated, if they are written as equations for the components of $-\boldsymbol{n}\cdot\nabla\boldsymbol{u}$. ($-\boldsymbol{n}\cdot\nabla\boldsymbol{u}$ is the gradient of $\boldsymbol{u}$ taken in direction of $-\boldsymbol{n}$, i.e. from the wall orthogonal into the interior of the gas.)

$$-\boldsymbol{n}\cdot\nabla\boldsymbol{u}\cdot\frac{\boldsymbol{u}}{u} = \frac{1}{2\,\zeta}\,u + \boldsymbol{\omega}\cdot\frac{\boldsymbol{u}\times\boldsymbol{n}}{u} \,, \tag{16}$$

$$-\boldsymbol{n}\cdot\nabla\boldsymbol{u}\cdot\frac{\boldsymbol{u}\times\boldsymbol{n}}{u} = -\boldsymbol{\omega}\cdot\frac{\boldsymbol{u}}{u} \,, \tag{17}$$

with

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\mathrm{curl}\,\boldsymbol{u}$$

as vorticity of the gas.

3. If the vorticity $\boldsymbol{\omega}$ was no $\boldsymbol{u}$-component,

$$\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{u} = 0 \,, \tag{18}$$

the second boundary condition (17) is simply

$$\boldsymbol{n}\cdot\nabla\boldsymbol{u}\cdot\frac{\boldsymbol{u}\times\boldsymbol{n}}{u} = 0 \,. \tag{19}$$

According to a theorem in differential geometry (18) is the necessary and sufficient condition that the flow lines $\boldsymbol{u}$ are orthogonal trajectories of a family U of surfaces. Such $\boldsymbol{u}$-fields are called quasi-potential fields. The following considerations are related to quasipotential fields.

We try to introduce a curvilinear coordinate system (ξ^1, ξ^2, ξ^3) (similarly as Waldmann [4] in his "Non-Equilibrium Thermodynamics of Boundary Conditions"). As first coordinate ξ^1 we take the familiy parameter of U. The surfaces of points having the same value of the coordinate ξ^2 are determined as follows. The wall itself is considered as one of the ξ^2-surfaces. The other ξ^2-surfaces are orthogonal to $(-\boldsymbol{n})$-lines lying on the surfaces of U and having the direction of $-\boldsymbol{n}$ at the wall. Finally, the ξ^3-surfaces are chosen to be orthogonal to $\boldsymbol{u}\times(-\boldsymbol{n})$-lines which lie also on the surfaces of the family U and are orthogonal to the $(-\boldsymbol{n})$-lines. The field of the tangent unit vectors of the $(-\boldsymbol{n})$-lines is called the $(-\boldsymbol{n})$-vector-field and that of the $\boldsymbol{u}\times(-\boldsymbol{n})$-lines the $\boldsymbol{u}\times(-\boldsymbol{n})$-vector-field. Basing on the two conditions that both the $(-\boldsymbol{n})$-field and the $(\boldsymbol{u}\times(-\boldsymbol{n}))$-field are quasipotential, together with condition (18) it follows that the geodesic torsion T_2 of the $(-\boldsymbol{n})$-lines is equal to zero

$$T_2 = 0 \,. \tag{20}$$

By the necessary condition (20) the $(-\boldsymbol{n})$-lines are uniquely determined on the surfaces of U, hence also the $(\boldsymbol{u}\times(-\boldsymbol{n}))$-lines. If the introduction of such a curvilinear coordinate system is possible, the second slip boundary condition (19) is fulfilled in the system (ξ^1, ξ^2, ξ^3), since

$$T_2 = -(-\boldsymbol{n}) \cdot \nabla \frac{\boldsymbol{u}}{u} \cdot \left(\frac{\boldsymbol{u}}{u} \times (-\boldsymbol{n}) \right).$$

The first slip boundary condition can be written in the system (ξ^1, ξ^2, ξ^3) by means of the transformation equation

$$\nabla \boldsymbol{u} = \sum_{i,1} \boldsymbol{e}^i \, \boldsymbol{e}_l \left[\frac{\partial u^l}{\partial \xi^i} + \sum_k \left\{ \begin{matrix} i\,k \\ l \end{matrix} \right\} u^k \right],$$

wherein $\boldsymbol{e}_l$ denotes the basis vectors of the coordinate system, $\boldsymbol{e}^i$ the basis vectors of the reciprocal system, u^l the (contravariant) coordinates of $\boldsymbol{u}$ with respect to the basis $\boldsymbol{e}_l$ and $\left\{ \begin{smallmatrix} i\,k \\ 1 \end{smallmatrix} \right\}$ the Christoffel symbols of the second kind. The basis vectors $\boldsymbol{e}_l$ are connected with the fundamental tensor g_{ik} owing to

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_k = g_{ik} \, .$$

For the special orthogonal coordinate system used here, it is

$$\boldsymbol{u} = u \, \boldsymbol{e}_1 / \sqrt{g_{11}} \, ,$$

$$-\boldsymbol{n} = \boldsymbol{e}_2 / \sqrt{g_{22}} \, ,$$

$$\boldsymbol{u} \times (-\boldsymbol{n}) = u \, \boldsymbol{e}_3 / \sqrt{g_{33}} \, ,$$

and the first slip boundary condition (14)

$$u = \zeta \sqrt{\frac{g_{11}}{g_{22}}} \frac{\partial}{\partial \xi^2} \left(\frac{u}{\sqrt{g_{11}}} \right).$$

Especially for the rotationally symmetric flow around a sphere, the condition is

$$u = \zeta \, r \frac{\partial}{\partial r} \left(\frac{u}{r} \right). \tag{21}$$

(21) gives a correction of order l/r (l: mean free path) compared with the slip boundary condition for plane surfaces. This correction has an essential influence in that region of slip flow regime lying near the transition flow regime.

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